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Some Cylindrical Symmetric Nonstatic Perfect Fluid Distributions in General Relativity with Pressure Equal to Density

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In this paper we have derived some models of cylindrical symmetry in which source of gravitational field is perfect fluid with pressure equal to energy density.

1. INTRODUCTION

Plane symmetric space-times representing distributions of perfect fluid with isentropic flow was discussed in detail by Taub (1956). Nonstatic solutions of plane symmetry are particularly interesting as they may represent galaxies which by and large exhibit plane symmetry. Tabensky and Taub (1973) have discussed plane-symmetric distributions of perfect fluid with irrotational flow which satisfy the equation of state $\rho=p$. This latter condition is satisfied in the case of a relativistic degenerate Fermi gas or in the case of neutron stars having very dense baryon matter (Zeldovich and Novikov, 1971; Walecka, 1974). Further work in this field was done by Letelier (1975, 1979), Letelier and Tabensky (1975), and Singh and Yadava (1978) in the case of cylindrical symmetry. In the present paper we have obtained a few solutions with the above equation of state in the case of the Marder metric which is of cylindrical symmetry.

We consider the metric in the form

$$ds^2=A^2(dt^2-dx^2)-B^2\,dy^2-c^2\,dz^2 \tag{1}$$

where the metric potentials are functions of x and t alone. The energy-momentum tensor for perfect fluid distribution is given by

$$T_{ij}=(\rho+p)v_iv_j-pg_{ij} \tag{2}$$

together with

$$g^{ij}v_i v_j = 1 \tag{3}$$

ρ being the density, p the pressure, and v_i the flow vector. The field equations to be satisfied are

$$-8\pi T_{ij} = R_{ij} - \tfrac{1}{2}Rg_{ij} \tag{4}$$

Equations (2) and (4) for metric (1) lead to

$$v_2 = v_3 = 0 \tag{5}$$

The field equations (4) lead to

$$-8\pi\left[(\rho+p)v_1^2 + pA^2\right] = \left[\frac{B_{44}}{B} + \frac{C_{44}}{C} - \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) - \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{B_1C_1 - B_4C_4}{BC}\right] \tag{6}$$

$$-8\pi pA^2 = \left[\left(\frac{A_4}{A}\right)_4 - \left(\frac{A_1}{A}\right)_1 + \frac{B_{44}}{B} - \frac{B_{11}}{B}\right] \tag{7}$$

$$-8\pi pA^2 = \left[\left(\frac{A_4}{A}\right)_4 - \left(\frac{A_1}{A}\right)_1 + \frac{C_{44}}{C} - \frac{C_{11}}{C}\right] \tag{8}$$

$$-8\pi\left[(\rho+p)v_4^2 - pA^2\right] = \left[\frac{B_{11}}{B} + \frac{C_{11}}{C} - \frac{A_1}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right) - \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{B_1C_1 - B_4C_4}{BC}\right] \tag{9}$$

$$-8\pi\left[(\rho+p)v_1v_4\right] = \left[\frac{B_{14}}{B} + \frac{C_{14}}{C} - \frac{A_1}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) - \frac{A_4}{A}\left(\frac{B_1}{B} + \frac{C_1}{C}\right)\right] \tag{10}$$

From equations (3), we have

$$v_4^2 - v_1^2 = A^2 \tag{11}$$

Suffixes 1 and 4 in the above equations after A, B, and C denote partial differentiation with respect to x and t, respectively.

2. SOLUTIONS OF THE FIELD EQUATIONS

From equations (7) and (8) we have

$$\frac{B_{uv}}{B}=\frac{C_{uv}}{C} \tag{12}$$

where

$$u=\tfrac{1}{2}(x+t)$$

$$v=\tfrac{1}{2}(x-t) \tag{13}$$

From equations (6), (9), and equation of state $\rho=p$ we have

$$2\frac{B_{uv}}{B}+\frac{B_uC_v+B_vC_u}{BC}=0 \tag{14}$$

From equations (6), (7), (9), (10), and (12) we have

$$\left[\left(\frac{B_{uu}}{B}+\frac{C_{uu}}{C}\right)-2\frac{A_u}{A}\left(\frac{B_u}{B}+\frac{C_u}{C}\right)\right]\times\left[\left(\frac{B_{vv}}{B}+\frac{C_{vv}}{C}\right)-2\frac{A_v}{A}\left(\frac{B_v}{B}+\frac{C_v}{C}\right)\right]$$

$$=\left[\frac{B_uC_v+B_vC_u}{BC}-2\left(\frac{A_u}{A}\right)_v\right]^2 \tag{15}$$

Case (a). Putting $BC=\mu$ and $B/C=\nu$, we get from equations (12) and (14), respectively,

$$2\left(\frac{\nu_u}{\nu}\right)_v+\frac{\mu_u\nu_v+\mu_v\nu_u}{\mu\nu}=0 \tag{16}$$

and

$$2\left(\frac{\mu_{uv}}{\mu}\right)+2\left(\frac{\nu_u}{\nu}\right)_v+\frac{\mu_u\nu_v+\mu_v\nu_u}{\mu\nu}=0 \tag{17}$$

From equations (16) and (17), we have

$$\mu_{uv}=0 \tag{18}$$

which leads to

$$\mu = f(u) + g(v) \tag{19}$$

Case (a.1). Let us assume that

$$\log \nu = \delta(u) + \theta(v) \tag{20}$$

From equations (16), (17), (19), and (20) we have

$$\frac{\delta_u}{f_u} = -\frac{\theta_v}{g_v} = a \tag{21}$$

where a is constant. Integrating equations (21) we have

$$\log \nu = a(f - g) + b_0 \tag{22}$$

where b_0 is a constant. Arriving at $A = A(\mu)$ from equation (15), we have

$$\left[2\mu\frac{f_{uu}}{f_u^2} - (1 - a^2\mu^2) - 4\mu\frac{d}{d\mu}(\log A)\right] \times \left[2\mu\frac{g_{vv}}{g_v^2} - (1 - a^2\mu^2) - 4\mu\frac{d}{d\mu}(\log A)\right]$$
$$= \left[(1 + a^2\mu^2) - 4\mu^2\frac{d^2}{d\mu^2}(\log A)\right]^2 \tag{23}$$

From equation (23) it is clear that (f_{uu}/f_u^2) and (g_{vv}/g_v^2) both should be constant. In particular we assume that

$$\frac{f_{uu}}{f_u^2} = \frac{g_{vv}}{g_v^2} = \lambda \tag{24}$$

where λ is a constant. From equations (19), (22), and (23) we have

$$B = \left[\frac{Q'}{\lambda}\left(\frac{V}{U}\right)^{a/\lambda} \log \frac{P}{UV}\right]^{1/2} \tag{25}$$

$$C = \left[\frac{1}{Q'\lambda}\left(\frac{U}{V}\right)^{a/\lambda} \log \frac{P}{UV}\right]^{1/2} \tag{26}$$

where $U=u+u_0$, $V=v+v_0$, u_0, v_0, P, and Q' are constants. From equations (23) and (24) we have

$$\left[2\mu\lambda-(1-a^2\mu^2)-4\mu\frac{d}{d\mu}(\log A)\right]=\pm\left[(1+a^2\mu^2)-4\mu^2\frac{d^2}{d\mu^2}(\log A)\right] \tag{27}$$

Two cases arise:

Case (a.1.1). Taking the upper sign in (27), we get

$$A=\left\{\frac{1}{\lambda}\log\frac{P}{UV}\right\}^{-1/4}\exp\left\{\frac{1}{2}\left[\xi\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^2+\log\frac{P}{UV}+Q\right]\right\} \tag{28}$$

where ξ and Q are constants. By suitable transformation the metric (1) reduces to the form

$$ds^2=\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^{-1/2}\exp\left[\xi\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^2+\log\frac{P}{UV}+Q\right](dT^2-dX^2)$$
$$-\log\frac{P}{UV}\left[\left(\frac{V}{U}\right)^{a/\lambda}dY^2+\left(\frac{U}{V}\right)^{a/\lambda}dZ^2\right] \tag{29}$$

where

$$U=\tfrac{1}{2}(X+T)$$
$$V=\tfrac{1}{2}(X-T) \tag{30}$$

Case (a.1.2). Taking the lower sign in (27), we get

$$A=\left[\frac{1}{\lambda}\log\frac{P}{UV}\right]^{\epsilon}\exp\left\{\frac{1}{2}\left[\frac{a^2}{4}\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^2+\log\frac{P}{UV}+q'\right]\right\} \tag{31}$$

where ϵ and q' are constants. By suitable transformation the metric (1) reduces to the form

$$ds^2=\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^{2\epsilon}\exp\left[\frac{a^2}{4}\left(\frac{1}{\lambda}\log\frac{P}{UV}\right)^2+\log\frac{P}{UV}+q'\right](dT^2-dX^2)$$
$$-\log\frac{p}{UV}\left[\left(\frac{V}{U}\right)^{a/\lambda}dY^2+\left(\frac{U}{V}\right)^{a/\lambda}dZ^2\right] \tag{32}$$

Case (a.2). Let us assume that

$$\mu=\mu(u) \tag{33}$$

From equations (16) and (33), we have

$$\log \nu=\mu^{-1/2}[F(u)+G(v)] \tag{34}$$

where F and G are arbitrary functions of u and v, respectively. From equations (15), (33), and (34) we have

$$\left[2\left(\frac{\mu_u}{\mu}\right)_u+\frac{\mu_u^2}{\mu^2}+\frac{\nu_u^2}{\nu^2}-4\frac{\mu_u}{\mu}\left(\frac{A_u}{A}\right)\right]\times\left[\frac{\nu_v^2}{\nu^2}\right]=\left[\frac{\nu_u\nu_v}{\nu^2}+4\left(\frac{A_u}{A}\right)_v\right]^2 \tag{35}$$

We assume that

$$\left[2\left(\frac{\mu_u}{\mu}\right)_u+\frac{\mu_u^2}{\mu^2}+\frac{\nu_u^2}{\nu^2}-4\frac{\mu_u}{\mu}\left(\frac{A_u}{A}\right)\right]=\left(\frac{\nu_v^2}{\nu^2}\right) \tag{36}$$

From equations (35) and (36), we have

$$4\left(\frac{A_u}{A}\right)_v+\frac{\nu_u\nu_v}{\nu^2}=\pm\left[\frac{\nu_v^2}{\nu^2}\right] \tag{37}$$

Case (a.2.1). Taking the upper sign in (37), we have

$$4\left(\frac{A_u}{A}\right)_v+\frac{\nu_u\nu_v}{\nu^2}=\frac{\nu_v^2}{\nu^2} \tag{38}$$

From equations (33), (34), (36), and (38), we obtain

$$\frac{\mu_u}{\mu}=-2\frac{G_{vv}}{G_v}=l \tag{39}$$

where l is a constant. Integrating equations (39) we have

$$\mu=e^{lu+l'} \tag{40}$$

$$G=-\frac{2}{l}e^{-(l/2)v+l''} \tag{41}$$

where l' and l'' are constants. Hence from equations (33), (34), (40), and

(41), we get

$$B=\frac{1}{a_1}\exp\left\{\frac{1}{2}\left[lu+a_1Fe^{-(l/2)u}-b_1e^{-(l/2)(v+u)}\right]\right\} \tag{42}$$

$$C=\frac{1}{a_1}\exp\left\{\frac{1}{2}\left[lu-a_1Fe^{-(l/2)u}+b_1e^{-(l/2)(v+u)}\right]\right\} \tag{43}$$

where $a_1=e^{-l'/2}$, $b_1=(2/l)e^{-l'/2+l''}$. From equations (36), (40), and (41), we get

$$4\log A=\left[lu+\frac{a_1^2}{l}M(u)+a_1b_1e^{-lv/2}N(u)+\log L(v)\right] \tag{44}$$

where

$$M(u)=\int e^{-lu}\left(F_u-\frac{l}{2}F\right)^2 du,\qquad N(u)=\int e^{-lu}\left(F_u-\frac{l}{2}F\right)du$$

and $L(v)$ is an arbitrary function of v. By suitable transformation the metric (1) reduces to the form

$$\begin{aligned} ds^2=&[L(v)]^{1/2}\exp\left\{\frac{1}{2}\left[lu+\frac{a_1^2}{l}M(u)+a_1b_1e^{-(l/2)v}N(u)\right]\right\}(dt^2-dx^2)\\ &-\exp\left[lu+a_1Fe^{-(l/2)u}-b_1e^{-(l/2)(v+u)}\right]dY^2\\ &-\exp\left[lu-a_1Fe^{-(l/2)u}+b_1e^{-(l/2)(v+u)}\right]dZ^2 \end{aligned} \tag{45}$$

Case (a.2.2). Taking the lower sign in (37), we have

$$4\left(\frac{A_u}{A}\right)_v+\frac{\nu_u\nu_v}{\nu^2}=-\frac{\nu_v^2}{\nu^2} \tag{46}$$

From equations (33), (34), (36) and (46) we have

$$\frac{\mu_u}{\mu}=2\frac{G_{vv}}{G_v}=K \tag{47}$$

where K is a constant. Integrating equations (47) we get

$$\mu = e^{Ku+K'} \tag{48}$$

$$G = \frac{2}{K} e^{Kv/2+K''} \tag{49}$$

where K' and K'' are constants. Hence from equations (33), (34), (48), and (49) we have

$$B = \frac{1}{p_0} \exp\left\{\frac{1}{2}\left[Ku + p_0 F e^{-Ku/2} + q_0 e^{K(v-u)/2}\right]\right\} \tag{50}$$

$$C = \frac{1}{p_0} \exp\left\{\frac{1}{2}\left[Ku - p_0 F e^{-Ku/2} - q_0 e^{K(v-u)/2}\right]\right\} \tag{51}$$

where $p_0 = e^{-K'/2}$, $q_0 = (2/K)e^{K''-K'/2}$. From equations (36), (48), and (50), we get

$$4\log A = \left[Ku + \frac{p_0^2}{K} J(u) - p_0 q_0 e^{Kv/2} H(u) + \log \dot{Q}(v)\right] \tag{52}$$

where

$$J(u) = \int e^{-Ku}\left(F_u - \frac{K}{2}F\right)^2 du, \qquad H(u) = \int e^{-Ku}\left(F_u - \frac{K}{2}F\right) du$$

and $Q(v)$ is an arbitrary function of v. By suitable transformation the metric (1) reduces to the form

$$\begin{aligned} ds^2 = {} & [Q(v)]^{1/2} \exp\left\{\frac{1}{2}\left[Ku + \frac{p_0^2}{K} J(u) - p_0 q_0 e^{Kv/2} H(u)\right]\right\}(dt^2 - dx^2) \\ & - \exp\left[Ku + p_0 F e^{-Ku/2} + q_0 e^{K(v-u)/2}\right] dY^2 \\ & - \exp\left[Ku - p_0 F e^{-Ku/2} - q_0 e^{K(v-u)/2}\right] dZ^2 \end{aligned} \tag{53}$$

Case (b). Let us assume that B and C are functions of A. Hence from equations (12) and (14), respectively, we have

$$\left[\frac{CB''-BC''}{CB'-BC'}\right]=-\left[\frac{A_{uv}}{A_u A_v}\right] \tag{54}$$

$$\left[\frac{B''}{B'}+\frac{C'}{C}\right]=-\left[\frac{A_{uv}}{A_u A_v}\right] \tag{55}$$

where a dash denotes differentiation with respect to A. Equating the left-hand side of equations (54) and (55), we get

$$B=a_0 C^{1/(a'-1)} \tag{56}$$

where a_0 and a' are constants. From equations (54) and (55), we conclude that

$$\frac{A_{uv}}{A_u A_v}=\psi(A) \tag{57}$$

From equation (57), we have

$$A=A\{\alpha(u)+\beta(v)\}=A(\chi) \tag{58}$$

where $\chi=\alpha+\beta$. Hence from equations (57) and (58) we have

$$\frac{A_{uv}}{A_u A_v}=\frac{\ddot{A}}{\dot{A}^2} \tag{59}$$

where a dot denotes differentiation with respect to χ. From equations (55) and (59) we have

$$\dot{B}=M_0/C \tag{60}$$

where M_0 is a constant. From equations (56) and (60) we get

$$B=a_0\phi^m \tag{61}$$

$$C=\phi^{1-m} \tag{62}$$

where

$$\phi = m'\chi + N'$$

$$m' = \frac{M_0 a'}{a_0}, \qquad m = \frac{1}{a'}$$

and N' is a constant.

Equations (15), (61), and (62) lead to

$$\left[2m(m-1) - 2\frac{\phi}{A}\left(\frac{dA}{d\phi}\right) + \frac{\phi}{m'}\left(\frac{\alpha_{uu}}{\alpha_u^2}\right)\right] \times \left[2m(m-1) - 2\frac{\phi}{A}\left(\frac{dA}{d\phi}\right) + \frac{\phi}{m'}\left(\frac{\beta_{vv}}{\beta_v^2}\right)\right] = 4\left[m(m-1) - \phi^2\frac{d}{d\phi}\left(\frac{1}{A}\cdot\frac{dA}{d\phi}\right)\right]^2 \tag{63}$$

From equation (63) it is clear that (α_{uu}/α_u^2) and (β_{vv}/β_v^2) both should be constant. In particular we assume that

$$\frac{\alpha_{uu}}{\alpha_u^2} = \frac{\beta_{vv}}{\beta_v^2} = b \tag{64}$$

where b is a constant. From equations (63) and (64) we have

$$\left[2m(m-1) - 2\frac{\phi}{A}\left(\frac{dA}{d\phi}\right) + n\phi\right] = \pm 2\left[m(m-1) - \phi^2\frac{d}{d\phi}\left(\frac{1}{A}\frac{dA}{d\phi}\right)\right] \tag{65}$$

where $n = b/m'$. Two cases arise:

Case (b.1). Taking the upper sign in (65), we have

$$A = \exp\left[\frac{1}{2}(q\phi^2 + n\phi + \gamma)\right] \tag{66}$$

where q and γ are constants. Also from equation (64) and $\phi = m'\chi + N'$ we have

$$\phi = K_0 - (1/n)\log(X^2 - T^2) \tag{67}$$

where

$$u + \eta = U = \tfrac{1}{2}(X + T), \qquad v + \xi = V = \tfrac{1}{2}(X - T)$$

and K_0, η, and ξ are constants. By suitable transformation the metric (1) reduces to the form

$$ds^2 = e^{(q\phi^2+n\phi+\gamma)}(dT^2 - dX^2) - \phi^{2m}\, dY^2 - \phi^{2(1-m)}\, dZ^2 \tag{68}$$

where ϕ is given by equation (67).

Case (b.2). Taking the lower sign in (65), we get

$$A = \phi^h \exp\left\{\tfrac{1}{2}\left[2m(m-1)(\log\phi)^2 + n\phi + \gamma_0\right]\right\} \tag{69}$$

where h and γ_0 are constants. By suitable transformation the metric (1) reduces to the form

$$ds^2 = \phi^{2h} e^{[2m(m-1)(\log\phi)^2 + n\phi + y_0]}(dT^2 - dX^2) - \phi^{2m}\, dY^2 - \phi^{2(1-m)}\, dZ^2 \tag{70}$$

3. SOME PHYSICAL AND GEOMETRICAL FEATURES

The expressions for pressure (density), flow vector, and reality conditions for different cases are as follows.

Case (a.1.1).

$$8\pi\rho = 8\pi p = \frac{(4P^2 - a^2)}{\lambda^2 A^2 (X^2 - T^2)} \tag{71}$$

$$v_1 = -\frac{AT}{(X^2 - T^2)^{1/2}} \tag{72}$$

$$v_4 = \frac{AX}{(X^2 - T^2)^{1/2}} \tag{73}$$

The reality condition requires that $4P^2 - a^2 > 0$. The space-time represented by the model is in general of Petrov type I; however, for $a = 0$ it is of type D.

Case (a.1.2).

$$8\pi\rho = 8\pi p$$

$$= \frac{1}{(A\log P/UV)^2}\left[\frac{-(4\epsilon+1)}{(X^2-T^2)}\right] \tag{74}$$

$$v_1 = -\frac{AX}{[-(X^2-T^2)]^{1/2}} \tag{75}$$

$$v_4 = \frac{AT}{[-(X^2-T^2)]^{1/2}} \tag{76}$$

It is clear that the model is realistic only when $X^2-T^2<0$. Hence $P<0$. The reality condition in this case requires that $(4\epsilon+1)>0$. For this model also space-time in general is of type I and it is type D for $a=0$.

Case (a.2.1).

$$8\pi\rho = 8\pi p$$

$$= \left[\frac{b_1^2 l^2}{16A^2}\right] e^{-l(v+u)} \tag{77}$$

$$v_1 = -A\left[1-\frac{2a_1}{b_1 l}e^{lv/2}\right]^{1/2} \tag{78}$$

$$v_4 = A\left[2-\frac{2a_1}{b_1 l}e^{lv/2}\right]^{1/2} \tag{79}$$

The reality condition for this case is identically satisfied. The model represents a space-time which is in general of Petrov type I. However, for $a_1=0$ or $F(u)=e^{lu/2+l_0}$ it is of type II and for $a_1=0$ and $L(v)=e^{4lv+l_0}$ it is of type D.

Case (a.2.2). This model turns out to be unrealistic.

Case (b.1).

$$8\pi\rho = 8\pi p$$

$$= \frac{4[q\phi^2+m(m-1)]}{n^2A^2\phi^2(X^2-T^2)} \tag{80}$$

$$v_1 = -\frac{A}{\sqrt{2}}\left[\frac{q\phi^2 - m(m-1)}{q\phi^2 + m(m-1)}\left(\frac{X^2+T^2}{X^2-T^2}\right) - 1\right]^{1/2} \tag{81}$$

$$v_4 = \frac{A}{\sqrt{2}}\left[\frac{q\phi^2 - m(m-1)}{q\phi^2 + m(m-1)}\left(\frac{X^2+T^2}{X^2-T^2}\right) + 1\right]^{1/2} \tag{82}$$

Since by (67) $X^2 - T^2 > 0$ the reality condition requires that $q\phi^2 + m(m-1) > 0$. The space-time represented by the model is in general of Petrov type I. However, for $m = \frac{1}{2}$ it is of type D.

Case (b.2).

$$8\pi\rho = 8\pi p = \frac{4[3m(m-1) - qm(m-1)\log\phi - h]}{n^2 A^2 \phi^2 (X^2 - T^2)} \tag{83}$$

$$v_1 = -\frac{A}{\sqrt{2}}\left[\frac{h + 2m(m-1)\log\phi - m(m-1)}{3m(m-1) - 2m(m-1)\log\phi - h}\left(\frac{X^2+T^2}{X^2-T^2}\right) - 1\right]^{1/2} \tag{84}$$

$$v_4 = \frac{A}{\sqrt{2}}\left[\left\{\frac{h + 2m(m-1)\log\phi - m(m-1)}{3m(m-1) - 2m(m-1)\log\phi - h}\right\}\left(\frac{X^2+T^2}{X^2-T^2}\right) + 1\right]^{1/2} \tag{85}$$

The reality condition in this case is

$$3m(m-1) - 2m(m-1)\log\phi - h > 0$$

The model represents a space-time, which is in general of Petrov type I. However, for $m = \frac{1}{2}$ it is of type D.

All the above models are rotating shearing expanding and nongeodetic in general.

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